REDUCED-FORM MECHANISM DESIGN AND EX POST FAIRNESS CONSTRAINTS

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Abstract. This paper incorporates *fairness constraints* into the classic single-unit reduced-form implementation problem (Border, 1991, 2007; Che, Kim, & Mierendorff, 2013; Manelli & Vincent, 2010) with two agents. To do so, I use a new approach that utilizes the results from Kellerer (1961) and Gutmann, Kemperman, Reeds, and Shepp (1991). Under realistic assumptions on the constraints, the conditions are transparent and can be verified in polynomial time.

1. Introduction

The traditional objectives of mechanism design include aggregate welfare maximization, profit maximization, and budget balance. However, it can be desirable to add some fairness constraints to this list, as well. In allocation mechanisms with money, fairness constraints restrict ex post allocations, and they may be carried out even if they undermine aggregate welfare maximization, profit maximization, or budget balance.[1] For example, many government procurement and allocation programs are required by law to favor small businesses (Pai & Vohra, 2012). In spectrum auctions, sellers may set allocation guarantees to prevent bidders with low bids from starvation and thus prevent them from dropping out of future auctions (Wu, Zhong, & Chen, 2014). Fairness constraints can also induce less variation in payments, which is desirable when agents have budget constraints (Sinha & Anastasopoulos, 2017).

In this paper, I study the two-agent feasible reduced-form problem when ex post allocation probabilities have type-contingent fairness constraints. My results can be used to study single item allocation problems, such as ex post welfare maximizing auctions with two buyers.

1.1. Feasible Reduced-Form Problem.

A selling mechanism allocates goods via money transfer. For example, an indivisible item is often awarded to the highest bidder in a single-unit auction. When the bidders have private values, the ex post allocation rule to n bidders is the joint winning probability $q = (q_1, .., q_n)$ given the type profile, $t = (t_1, .., t_n)$, such that

$$q_i(t) \geq 0, i = 1, ..., n, \quad \sum_{j=1}^{n} q_j(t) \leq 1.$$

From prior work, such as that done by Myerson (1981), we understand that a Bayesian Nash equilibrium for quasi-linear utility bidders can be completely specified with the *reduced form*, which is a vector $Q = (Q_1, ..., Q_n)$ of the interim allocations:

$$Q_i(t_i) = E_{t_{-i}}(q_i(t)), i = 1, ..., n.$$

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[1]When money transfer is restricted, randomization is often used to guarantee ex ante fairness. So, for example, suppose a mother has a treat to give to only one of her two children, and she is indifferent as to which child gets the treat. She likely prefers a lottery that gives each child an equal probability of receiving the treat over a deterministic allocation to either child (see Machina, 1989). In practice, random serial dictatorship (Hylland & Zeckhauser, 1979) and the probabilistic serial procedure (Bogomolnaia & Moulin, 2001) are examples of ex ante fair assignment mechanisms. Additionally, Budish, Che, Kojima, and Milgrom (2013) discussed the implementation of a multiunit stochastic mechanism with integer capacity constraints and quotas by lotteries.

Interim allocations have much smaller dimensions than ex post allocations, so a reduced form is useful in mechanism design problems. However, a reduced form is *implementable* (or equivalently, is *feasible*) if and only if there exists a corresponding ex post allocation rule.

Maskin and Riley (1984) first discussed and proved a special case of the feasible reduced-form problem in optimal selling mechanisms. Moreover, Matthews (1984) first proposed a conjecture on feasible reduced-form auctions: suppose all n agents draw their valuations (types) from the same type space, T. A symmetric interim allocation $Q : T \to [0,1]$ is implementable with $q_i : T^n \to [0,1]$ if and only if for every Borel set $A \subseteq T$,

$$n \int_A Q(x) dF(x) \leq 1 - \left[\int_{T \setminus A} dF(x) \right]^n . \tag{1.1}$$

Border (1991) first proved conjecture (1.1) for symmetric auctions in a single-unit environment. Later, Border (2007) revisited this problem and generalized (1.1) for asymmetric auctions in finite type spaces. Mierendorff (2011) then extended Border's (1991) proof to asymmetric auctions. Later, Hart and Reny (2015) showed that the symmetric Border's theorem is equivalent to a second-order stochastic dominance condition with reference to an "efficient auction."

A fruitful method to generalize Border's theorem is to formulate the feasible reduced-form problem as a feasible circulation flow problem. Che et al. (2013) first used the circulation flow approach to study the reduced-form problem in multiunit auctions in which there are *paramodular set constraints* on the ex post allocation for any buyer subgroups.[2] Later, Li (2019) formulated a related circulation flow problem, providing the existence condition for the single-unit feasible reduced-form problem where the sum of ex post allocations across agents has a symmetric type-contingent lower bound.[3]

A related body of literature has focused on the equivalence of Bayesian and dominant strategy incentive compatibility (BIC-DIC equivalence). By generalizing the geometric technique found in Border (1991), Manelli and Vincent (2010) proved that in the environment of single-unit, one-dimensional continuous private values and with linear utility, given any Bayesian incentive compatible (BIC) mechanism, there is a dominant strategy incentiev compatible (DIC) ex post allocation with the same interim allocation. Gershkov, Goeree, Kushnir, Moldovanu, and Shi (2013) proved a different notion of BIC-DIC equivalence in social choice settings by showing that a DIC ex post allocation is one with the smallest weighted Euclidean norm among all ex post allocations associated with the same BIC mechanism. Kushnir and Liu (2019) further extended this BIC-DIC equivalence result to nonlinear utilities to accommodate applications in principal-agent models.

1.2. **Contributions to the Literature.** The present paper investigates the feasible reduced-form problem for $n = 2$ when the ex post allocations are constrained by bounded integrable functions that depend on both agent types (see Theorems 2.1 and 2.2), interpreted as fairness constraints. Corollary 2.3 imposes realistic conditions on the constraints, making the conditions verifiable in polynomial time. Example 1 illustrates such a contribution.

Example 1. *Suppose two agents have interim allocations*

$$Q_1 = x^{\frac{1}{2}}, \quad Q_2 = \frac{2}{3} y, \quad x, y \in [0,1].$$

[2] Che et al. (2013) investigated the feasible circulation flow problem where the demand nodes are the interim allocations and the supply nodes are the ex post allocations. The paramodular constraints on the ex post allocations are incorporated into the flow capacity constraints from the supply nodes. The researchers drew upon a result from Hassin (1982) regarding the existence of feasible circulation flow with paramodular constraints to obtain the corresponding existence condition for ex post allocations. The paramodular constraints studied by Che et al. (2013) can be examined using polymatroid optimization, which is solvable using a greedy procedure (see Vohra, 2011, in particular the discussion in section 6.2).

[3] In her formulation, the demand node are the pairs $(t_i, i) \in (T, I)$, and the supply nodes are vectors $t \in T^n$, where the set of agents $I = \{1, ..., n\}$ share the same type space, T. The flow capacity from a supply node is a symmetric type-contingent function, $\rho(t)$. In the formulation, the constraints on the flow capacities in the network are also paramodular. Li (2019) obtained the existence condition of the reduced-form problem in which the sum of the symmetric ex post allocation is lower-bounded by $\rho(t)$.

The types are uniformly distributed. We then ask whether there is an ex post allocation that satisfies $1 - g_2(x,y) \le q_1(x,y) \le g_1(x,y), 1 - g_1(x,y) \le q_2(x,y) \le g_2(x,y)$ *for arbitrary* $g_1, g_2 : [0,1]^2 \to [0,1]$.

When $g_1 = g_2 = 1$, the existing literature (e.g. Border, 2007; Che et al., 2013; Manelli & Vincent, 2010) suggests that the answer is positive. However, the existing studies provide no answer for when $g_1, g_2 \in L^1([0,1]^2)$.

By Corollary 2.3, the answer is negative when $g_1 = y, g_2 = x^{\frac{1}{3}}$. However, the answer is positive when

$$g_1 = \begin{cases} y^{1/5} + 0.25 & 0 \le y < 0.75; \\ y^{1/4} & 0.25 \le y \le 1. \end{cases} \quad g_2 = \begin{cases} x + 0.25 & 0 \le x < 0.75; \\ x & 0.25 \le x \le 1. \end{cases} \tag{1.2}$$

Furthermore, Corollary 2.3 suggests that there is a DIC ex post allocation.

More generally, we solve Problem 2.1 in a nonatomic probability space $([0,1]^2, \mathcal{B}, \mu_1 \times \mu_2)$ (Theorem 2.1). When types are uniformly distributed, we solve Problem 2.2 with incentive compatibility constraints (Theorem 2.2 and Corollary 2.3). We then extend Corollary 2.3 to some special cases when the item is not necessarily sold (Corollaries 2.4 and 2.5).

Theorem 2.2 cannot be obtained from Theorem 2.1 combined with the incentive compatibility results found in Manelli and Vincent (2010), since the equivalence results do not apply to cases with type-contingent bounds. Nevertheless, when $g_1 = g_2 = 1$, our results become special cases of the existing results. For example, when $g_1 = g_2 = 1$, Theorem 2.1 follows from Che et al. (2013, Corollary 6). By combining Corollary 6 in Che et al. (2013) with Theorem 2 in Manelli and Vincent (2010) and setting $n = 2$, we obtain Theorem 2.2 with $g_1 = g_2 = 1$.

To prove the main results of Theorem 2.1 and Theorem 2.2, I reduce the problems to ones that are solvable by Lemma 3.1 and 3.2, respectively. Although Lemma 3.2 follows from Theorem 4 and 7 in Gutmann et al. (1991), Gutmann et al.'s Theorem 4 relies on a result found in Kellerer (1961), but without a proof for it. For completeness, I include simplified proofs of the relevant results from Kellerer (1961) and Gutmann et al. (1991) in the present paper.[4]

The remainder of this paper is structured as follows. Section 2 provides the main implementation results and their corollaries. Section 3 proves Theorems 2.1 and 2.2 based on Lemmas 3.1 and 3.2, respectively, and also proves Corollary 2.3. Section 4 proves Lemma 3.1. Finally, section 5 proves Lemma 3.2.

2. Main Results

To capture the most general fairness considerations, one might contemplate ex post allocations that are constrained by arbitrary bounded functions of both agents' valuations.

2.1. **Primitives.** In what follows, we consider a selling mechanism whereby there is one indivisible item to allocate to two agents. Each agent $i = 1, 2$ has a private value on $[0,1]$ distributed according to a nonatomic probability measure, μ_i. The joint type space $[0,1]^2$ is endowed with the product measure $\mu_1 \times \mu_2$.

The agents' interim allocations are given as $Q_1 : [0,1] \to [0,1]$ and $Q_2 : [0,1] \to [0,1]$, where $\int_0^1 Q_1 d\mu_1 + \int_0^1 Q_2 d\mu_2 \le 1$. We are interested in determining whether Q_1, Q_2 can be implemented by ex post allocations $q_1, q_2 : [0,1]^2 \to [0,1]$, constrained by any pair of type-contingent floor and ceiling constraints

$$l_i(x,y) \le q_i(x,y) \le g_i(x,y), \quad x, y \in [0,1]^2, i = 1, 2, \tag{2.1}$$

where $g_i, l_i : [0,1]^2 \to [0,1]$ are arbitrary integrable functions.

[4]Lemmas 3.2 and 3.1 are related to the results from the classic question regarding the existence of a zero-one matrix with given marginals, which is a special case of the integral version of the supply-demand theorem discussed by Gale (1957). Theorem 2.2 in Brualdi (1980) (see also Mirsky, 1968) is a similar form as Lemma 3.2 in this paper. However, Theorem 2.2 in Brualdi (1980) is stated for the integral components, while here, Lemma 3.2 is stated for any real numbers. Integers imply a corresponding result with rational components; perhaps one can also get arbitrary real numbers, but this does not seem to follow immediately (especially because – since there are many equalities in the assumption of the theorem – one needs to carefully approximate real numbers using rational ones to keep the constraints in the same direction).

2.1.1. *Maximal Auctions.* In the main theorems, we consider the following case:

$$q_1(x,y) + q_2(x,y) = 1 \quad \forall x, y \in [0,1]. \tag{2.2}$$

Suppose the seller maximizes the following utilitarian welfare function where the two agents are given equal weights:

$$xq_1(x,y) + yq_2(x,y), (x,y) \in [0,1]^2.$$

The seller does not care about budget balance, and thus, she can always arrange an individual rational money transfer. In this case, $q_1 + q_2 = 1$ holds for all direct mechanisms that are incentive compatible and individual rational.[5] Following Hart and Reny (2015), an auction where $q_1 + q_2 = 1$ is called a *maximal auction*. We consider nonmaximal auctions ($0 \le q_1 + q_2 \le 1$) in Corollaries 2.4 and 2.5.

2.1.2. *Ex Post Constraints.* We require that

$$l_1(x,y) = 1 - g_2(x,y), \quad l_2(x,y) = 1 - g_1(x,y) \tag{2.3}$$

for all $(x,y) \in [0,1]^2$.

(2.3) is necessary because we cannot rule out the case $q_i(x,y) = l_i(x,y)$ or $q_i(x,y) = g_i(x,y)$, $i = 1,2$ in our solutions to the problems posed. Suppose $l_1(x,y) < 1 - g_2(x,y)$ for some $(x,y) \in [0,1]^2$. Then, when $q_1(x,y) = l_1(x,y), q_1(x,y) + q_2(x,y) < 1 - g_2(x,y) + g_2(x,y) = 1$, this would contradict (2.2). Suppose $l_1(x,y) > 1 - g_2(x,y)$ for some $(x,y) \in [0,1]^2$. Then, when $q_1(x,y) = l_1(x,y), q_2(x,y) = g_2(x,y)$, $q_1(x,y) + q_2(x,y) > 1$, this would also contradict (2.2). Therefore, $l_1 = 1 - g_2$. Similarly, $l_2 = 1 - g_1$.

Finally, by $0 \le l_1(x,y) \le g_1(x,y) \le 1$ and $0 \le l_2(x,y) \le g_2(x,y) \le 1$ for all $(x,y) \in [0,1]^2$, we have $0 \le 1 - g_2 \le g_1 \le 1$ and $0 \le 1 - g_1 \le g_2 \le 1$, so

$$g_1(x,y) + g_2(x,y) \ge 1 \quad \forall (x,y) \in [0,1]^2.$$

2.2. Main Results.

Problem 2.1. *One is given two arbitrary, measurable functions $Q_1, Q_2 : [0,1] \mapsto [0,1]$. The goal is to obtain the necessary and sufficient assumptions for Q_1 and Q_2 that would guarantee the existence of the measurable functions q_1, q_2 defined on $[0,1] \times [0,1]$ with the following properties:*

(i) $\int_0^1 q_1(x,t)d\mu_2(t) = Q_1(x)$, *and* $\int_0^1 q_2(s,y)d\mu_1(s) = Q_2(y)$ *for a.e.* $x, y \in [0,1]$;
(ii) $q_1(x,y) + q_2(x,y) = 1$ *for a.e.* $x, y \in [0,1]$;
(iii) $1 - g_2(x,y) \le q_1(x,y) \le g_1(x,y)$, *and* $1 - g_1(x,y) \le q_2(x,y) \le g_2(x,y)$, *where* $0 \le g_1, g_2 \le 1$ *and* $g_1 + g_2 \ge 1$, $g_1, g_2 \in L^1([0,1]^2, \mu_1 \times \mu_2)$ *are given.*

Theorem 2.1 completely solves Problem 2.1.

Theorem 2.1. *The pair (q_1, q_2) that satisfies Problem 2.1 exists if and only if*

$$\int_0^1 Q_1 d\mu_1 + \int_0^1 Q_2 d\mu_2 = 1, \tag{2.4}$$

and

$$\mu_1(U) - \iint_{U \times V^c} g_2 d(\mu_1 \times \mu_2) \le \int_U Q_1 d\mu_1 + \int_V Q_2 d\mu_2 \le \iint_{U \times V^c} g_1 d(\mu_1 \times \mu_2) + \mu_2(V) \tag{2.5}$$

holds for all $U \times V \subseteq [0,1]^2$.

Notice that (2.4) and (2.5) imply the following constraint as well:

$$1 - \iint_{U^c \times V} g_1 d(\mu_1 \times \mu_2) - \mu_2(V^c) \le \int_U Q_1 d\mu_1 + \int_V Q_2 d\mu_2 \le 1 - \mu_1(U^c) + \iint_{U^c \times V} g_2 d(\mu_1 \times \mu_2)$$

The above constraint can be obtained simply by applying (2.5) to the set $U^c \times V^c$. Notice that (2.4) implies $\int_U Q_1 d\mu_1 + \int_V Q_2 d\mu_2 + \int_{U^c} Q_1 d\mu_1 + \int_{V^c} Q_2 d\mu_2 = 1$.

[5]An incentive compatible, individual rational, welfare-maximizing direct mechanism, where $q_i \in \{0,1\}, i = 1,2$, is established in Börgers and Krahmer (2015, Proposition 3.5).

2.3. **Incentive Compatibility with Uniform Types.** A mechanism is incentive compatible if truth-telling is an equilibrium strategy. Results found in Myerson (1981) suggests the following definition.[6]

Definition. *a mechanism is*

(1) *BIC if and only if the interim allocation is nondecreasing; while it is*
(2) *DIC if and only if a bidder's ex post allocation is nondecreasing in her type.*

Manelli and Vincent (2010) discussed the equivalence of BIC and DIC implementations without considering any ex post allocation constraints. We cannot directly apply such a result to Theorem 2.1 because we do not know if the ex post allocations still satisfy the ex post fairness constraints.

I have a characterization for incentive compatible implementations when the types are uniformly distributed, i.e., when $\mu_1 = \mu_2 = \lambda$, the Lebesgue measure. However, I have no corresponding result for nonuniform types.

Problem 2.2 (Incentive Compatible Implementation)**.** *One is given two arbitrary, nondecreasing, measurable functions $Q_1, Q_2 : [0,1] \mapsto [0,1]$. The goal is to obtain the necessary and sufficient assumptions for Q_1 and Q_2 that would guarantee the existence of the measurable functions q_1, q_2 defined on $[0,1] \times [0,1]$ which satisfies conditions (i), (ii), (iii) in Problem 2.1, and*

(iv) *The maps $x \mapsto q_1(x,t)$, and $y \mapsto q_2(s,y)$ are nondecreasing for a.e. $s,t \in [0,1]$.*

Further, $\mu_1 = \mu_2 = \lambda$.

The following theorem completely solves Problem 2.2.

Theorem 2.2. *The pair (q_1, q_2) that satisfies Problem 2.2 exists if and only if*

$$\int_0^1 (Q_1 + Q_2) = 1, \tag{2.6}$$

and

$$|U| - \iint_{(1-U)\times V^c} g_2 \le \int_U Q_1 + \int_V Q_2 \le \iint_{U\times(1-V)^c} g_1 + |V| \tag{2.7}$$

holds for all $U \times V \subseteq [0,1]^2$, where $1 - U = \{x \in [0,1] : x = 1 - y, y \in U\}$.

To further simplify the condition, we consider the case when one's allocation upper bound would be a nonincreasing function of the other's type alone, and one's allocation lower bound would be a nondecreasing function of one's own type alone.

Problem 2.3. *Problem 2.2 with (iii) replaced by*

(iii') $1 - g_2(x) \le q_1(x,y) \le g_1(y)$, *and* $1 - g_1(y) \le q_2(x,y) \le g_2(x)$. $g_1, g_2 \in L^1([0,1])$ *are given, they are nonincreasing with the properties* $0 \le g_1, g_2 \le 1$, $g_1 + g_2 \ge 1$.

Corollary 2.3. *The pair (q_1, q_2) that satisfies Problem 2.3 exists if and only if*

$$\int_0^1 (Q_1 + Q_2) = 1 \tag{2.8}$$

and

$$\int_0^s Q_1 + \int_0^t Q_2 \ge \max\{t - (1-s)\int_{1-t}^1 g_1, \quad s - (1-t)\int_{1-s}^1 g_2\} \quad \forall s,t \in [0,1]. \tag{2.9}$$

Condition 2.9 can be checked quickly. The corresponding algorithm used to test condition (2.9) for a discretized domain has polynomial complexity $O(N^2 n)$.[7]

[6] Standard proofs can be found, for example, in Börgers and Krahmer (2015, Propositions 3.2 and 7.2). Since money transfer can be recovered from the corresponding incentive compatible allocations, it is omitted here in the definition of an incentive compatible mechanism. See also Manelli and Vincent (2010, Definition 3.1).

[7] In the algorithm, an $N \times N$ grid is created, and one checks all points (s,t) on the grid. Take any point on the grid, (s,t). Functions g_1, g_2, Q_1, and Q_2 are all one-dimensional and integrable on the compact domain $[0,1]$ so they can be integrated using numerical integration methods that have polynomial time complexity $O(n)$, with n denoting

2.4. **Nonmaximal Auctions.** We provide some sufficient conditions for the generalization of Problem 2.3 below.

Problem 2.4. *Problem 2.3 with* (ii) *replaced by*

(ii') $0 \le q_1(x,y) + q_2(x,y) \le 1$ *for a.e.* $x, y \in [0,1]$.

Notice that (ii') holds if and only if $0 \le \int_0^1 Q_1 + Q_2 \le 1$. If it happens that $Q_1 + Q_2 \le 1$, then the problem has a trivial solution: $q_1(x,y) = Q_1(x)$ and $q_2(x,y) = Q_2(y)$. (Of course, the integral inequality $\int_0^1 (Q_1 + Q_2) \le 1$ does not necessarily mean that the integrand is, at most, 1.) This demonstrates that if Q_1, Q_2 are small enough, then the problem has a trivial positive solution; on the other hand, condition (2.9) requires Q_1, Q_2 to be large enough. Thus, maximal auctions and nonmaximal auctions can correspond to two completely opposite scenarios. However, we can extend Corollary 2.3 to some special cases of nonmaximal auctions.

Corollary 2.4. *Problem 2.4 has a positive solution if (1)* $\max\{\|Q_1\|_\infty, \|Q_2\|_\infty\} \le \int_0^1 (Q_1 + Q_2)$, *(2)* $g_i + l_j = \int_0^1 Q_1 + Q_2, i, j = 1, 2, i \ne j$, *and if (3)*

$$\int_0^s Q_1 + \int_0^t Q_2 \ge \max\{t \int_0^1 (Q_1 + Q_2) - (1-s) \int_{1-t}^1 g_1, \quad s \int_0^1 (Q_1 + Q_2) - (1-t) \int_{1-s}^1 g_2\} \tag{2.10}$$

holds for all $s, t \in [0,1]$.

Proof of Corollary 2.4. Note that if $\int_0^1 (Q_1 + Q_2) = 0$, then there is nothing to prove (i.e., taking $q_1 = q_2 = 0$ would suffice); otherwise, we can assume $\int_0^1 (Q_1 + Q_2) = k > 0$. Next, we can consider new functions: $\tilde{Q}_1 := Q_1/k$ and $\tilde{Q}_2 := Q_2/k$. By assumption (1) in the corollary, $\tilde{Q}_1, \tilde{Q}_2 : [0,1] \mapsto [0,1]$, $\tilde{Q}_1, \tilde{Q}_2$ are bounded, measurable, and nondecreasing, and also $\int_0^1 (\tilde{Q}_1 + \tilde{Q}_2) = 1$. Take $\tilde{g}_j = g_j/k, \tilde{l}_j = l_j/k, j = 1, 2$. Assumption (2) implies that they satisfy $\tilde{l}_2 = 1 - \tilde{g}_1$, $\tilde{l}_1 = 1 - \tilde{g}_2$.

Note that condition (2.10) implies

$$\int_0^s \tilde{Q}_1 + \int_0^t \tilde{Q}_2 \ge \max\{t - (1-s) \int_{1-t}^1 \tilde{g}_1, \quad s - (1-t) \int_{1-s}^1 \tilde{g}_2\} \quad \forall s, t \in [0,1].$$

Therefore, applying Corollary 2.3, we find functions $\tilde{q}_1 : [0,1]^2 \mapsto [1 - \tilde{g}_2, \tilde{g}_1]$ and $\tilde{q}_2 : [0,1]^2 \mapsto [1 - \tilde{g}_1, \tilde{g}_2]$ that satisfy Problem 2.3 conditions (i), (ii), (iii'), and (iv), where Q_j is replace by $\tilde{Q}_j$ and g_i is replaced by $\tilde{g}_j$ for $j = 1, 2$. Finally, note that the new pair, q_1, q_2, where $q_j := k\tilde{q}_j$ for $j = 1, 2$, satisfy problem 2.4 conditions (i), (ii'), (iii'), and (iv). ☐

Example 2. *Take* $Q_1 = \alpha x^{\frac{1}{2}}, Q_2 = \frac{2}{3}\alpha y, x, y \in [0,1], \alpha \in (0,1]$. *By Corollary 2.4, there exists a DIC ex post allocation that satisfies* $0 \le q_1 \le g_1, 0 \le q_2 \le g_2$ *when*

$$g_1 = \begin{cases} \alpha(y^{1/5} + 0.25) & 0 \le y < 0.75; \\ \alpha y^{1/4} & 0.25 \le y \le 1. \end{cases} \quad g_2 = \begin{cases} \alpha(x + 0.25) & 0 \le x < 0.75; \\ \alpha x & 0.25 \le x \le 1. \end{cases}$$

Corollary 2.5. *Problem 2.4 has a positive solution if (1)* $0 < \|Q_2\|_\infty \le \frac{\int_0^1 Q_2}{1 - \int_0^1 Q_1}$, *(2)* $l_1 = 1 - \frac{1 - \int_0^1 Q_1}{\int_0^1 Q_2} g_2, l_2 = \frac{\int_0^1 Q_2}{1 - \int_0^1 Q_1}(1 - g_1)$, $g_1 + \frac{\int_0^1 Q_2}{1 - \int_0^1 Q_1} g_2 \ge 1$, *and if (3)*

$$\int_0^s Q_1 + \int_0^t \frac{1 - \int_0^1 Q_1}{\int_0^1 Q_2} Q_2 \ge \max\{t - (1-s) \int_{1-t}^1 g_1, \quad s - (1-t) \int_{1-s}^1 \frac{1 - \int_0^1 Q_1}{\int_0^1 Q_2} g_2\} \tag{2.11}$$

holds for all $s, t \in [0,1]$.

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the number of integration points. The comparison operation to find $\max\{t - (1-s) \int_{1-t}^1 g_1, s - (1-t) \int_{1-s}^1 g_2\}$ and the comparison of the left-hand side and right-hand side of the inequality in (2.9) both have complexity $T(1)$. Thus, the time complexity to check the inequality in (2.9) for each (s,t) on the grid is $O(n)$. Therefore, the overall complexity to check (2.9) on the grid is $O(N^2 n)$.

Proof of Corollary 2.5. Take $k = \frac{\int_0^1 Q_2}{1-\int_0^1 Q_1}$. When $\int_0^1 Q_2 = 0$, then $Q_2 = 0$ a.e. In this case, $q_1 = 0, q_2 = Q_2$ would solve the problem. Next, consider the nontrivial case where $\int_0^1 Q_2 > 0$. Let $\tilde{Q}_2 = \frac{1}{k}Q_2$. Then, $\int_0^1 Q_1 + \int_0^1 \tilde{Q}_2 = 1$. By assumption (1), $\tilde{Q}_2 : [0,1] \to [0,1]$. Take $\tilde{l_2} = \frac{1}{k}l_2, \tilde{g_2} = \frac{1}{k}g_2$. By assumption (2), $l_1 = 1 - \tilde{g}_2, \tilde{l}_2 = 1 - g_1$.

Note that condition (2.11) implies

$$\int_0^s Q_1 + \int_0^t \tilde{Q}_2 \geq \max\{t-(1-s)\int_{1-t}^1 g_1, \quad s-(1-t)\int_{1-s}^1 \tilde{g}_2\} \quad \forall s,t \in [0,1].$$

Therefore, applying Corollary 2.3, we find functions $q_1, \tilde{q}_2 : [0,1]^2 \mapsto [0,1]$ that satisfy Problem 2.3 conditions (i), (ii), (iii'), and (iv), where Q_2 is replaced by $\tilde{Q}_2$. Finally, note that the new pair $q_1, q_2 = k\tilde{q_2}$ satisfy problem 2.4 conditions (i), (ii'), (iii'), and (iv) □

It can easily be checked that Corollary 2.5 also implies Q_1, Q_2, g_1, g_2 in Example 2 are feasible.

3. Proof of Theorem 2.1, Theorem 2.2, and Corollary 2.3

In this section, we reduce Theorem 2.1 to a problem solvable by Lemma 3.1, and we reduce Theorem 2.2 to a problem solvable by Lemma 3.2. We show Corollary 2.3 by further simplifying Theorem 2.2. The proof of Lemma 3.1 is found in section 4, and the proof of Lemma 3.2 is found in section 5.

First, we show that $\int_0^1 Q_1 + Q_2 = 1$ is necessary and sufficient for $q_1 + q_2 = 1$ for all $(x,y) \in [0,1]^2$. By integrating the inequality $0 \leq q_1 + q_2 \leq 1$ over the domain $[0,1]^2$, one gets

$$\int_0^1\int_0^1 (q_1(x,y)+q_2(x,y))dxdy = \int_0^1 Q_1(x)dx + \int_0^1 Q_2(y)dy = \int_0^1 (Q_1+Q_2).$$

Since all functions are nonnegative, it follows that we must have

$$\int_0^1 (Q_1+Q_2) = 1 \iff q_1+q_2 = 1 \quad \text{for all} \quad (x,y) \in [0,1]^2. \tag{3.1}$$

The same argument carries over to arbitrary nonatomic probability spaces $([0,1]^2, \mathcal{B}([0,1]^2), \mu_1 \times \mu_2)$.

3.1. Proof of Theorem 2.1. Step 1: Reduction to one function.

By (i) and (ii) in Problem 2.1, we have the equality $Q_2(y) = \int_0^1 q_2(x,y)d\mu_1(x) = 1 - \int_0^1 q_1(x,y)d\mu_1(x)$. Since $q_1 + q_2 = 1$, if we find $q_1 : [0,1]^2 \mapsto [0,g_1]$, then $q_2 \geq 1 - g_1$. Similarly, if we find $q_2 : [0,1]^2 \mapsto [0,g_2]$, then $q_1 \geq 1-g_2$. Thus, to satisfy Problem 2.1 (iii), it is necessary and sufficient to find $q_1 : [0,1]^2 \mapsto [0,g_1]$ and $q_2 : [0,1]^2 \to [0,g_2]$.

Take $\tilde{Q}_2(y) = 1 - Q_2(y)$. Clearly, $\tilde{Q}_2 : [0,1] \mapsto [0,1]$ is bounded and measurable. The condition $\int_0^1 Q_1 d\mu_1 + \int_0^1 Q_2 d\mu_2 = 1$ holds if and only if $\int_0^1 Q_1 d\mu_1 = \int_0^1 \tilde{Q}_2 d\mu_2$. Therefore, to find the pair (q_1, q_2) is the equivalent of finding the pair $f^1 : [0,1] \mapsto [0,g_1], f^2 :\mapsto [0,g_2]$, such that

$$\int_0^1 f^1(x,y)d\mu_2(y) = Q_1 =: f_1^1(x), \quad \int_0^1 f^1(x,y)d\mu_2(x) = 1 - Q_2(y) =: f_2^1(y) \tag{3.2}$$

for a.e. $x, y \in [0,1]$. All that is known about the functions $f_1^1, f_2^1 : [0,1] \mapsto [0,1]$ is that they are measurable with $\int_0^1 f_1 = \int_0^1 f_2$. Clearly, if such an f^1 exists, then $q_1(x,y) = f(x,y)$ and $q_2(x,y) = 1 - f(x,y)$ solves Problem 2.1; and conversely, if Problem 2.1 has a solution, then $f^1(x,y) = q_1(x,y)$ solves (3.2).

Applying the same argument, finding the required $q_2 : [0,1]^2 \to [0,g_2]$ is the equivalent of finding f^2, such that

$$\int_0^1 f^2(x,y)d\mu_2(y) = 1 - Q_1(x) =: f_1^2(x), \quad \int_0^1 f^2(x,y)d\mu_1(x) = Q_2 =: f_2^2(y) \tag{3.3}$$

for a.e. $x, y \in [0,1]$.

Step 2: Derive (2.5) **using Lemma 3.1.** Take $f_1, f_2 : [0,1] \mapsto [0,1]$, where f_1, f_2 are measurable in the spaces $([0,1], \mathcal{B}([0,1]), \mu_1)$ and $([0,1], \mathcal{B}([0,1]), \mu_2)$, respectively, and where $\int_0^1 f_1 d\mu_1 = \int_0^1 f_2 d\mu_2$. The goal is to understand what additional assumptions of f_1, f_2 guarantee the existence of $f \in L^1\left(\mu_1 \times \mu_2, [0,1]^2\right)$, such that

$$0 \le f \le g, g \in L^1([0,1], \mu_1 \times \mu_2) \text{ is bounded and nonnegative;} \tag{3.4a}$$

$$\int_0^1 f(x,y) d\mu_2(y) = f_1(x); \tag{3.4b}$$

$$\int_0^1 f(x,y) d\mu_1(x) = f_2(y) \tag{3.4c}$$

for almost every $x, y \in [0,1]$.

Lemma 3.1. *The necessary and sufficient condition for the existence of such an f in* (3.4) *is*

$$\int_U f_1 d\mu_1 \le \iint_{U \times V} g d(\mu_1 \times \mu_2) + \int_{V^c} f_2 d\mu_2 \textit{ for all measurable } U, V \subset [0,1]. \tag{3.5}$$

Applying Lemma 3.1 to conditions 3.2 and 3.3, we obtain:

$$\int_U Q_1 d\mu_1 \le \iint_{U \times V} g_1 d(\mu_1 \times \mu_2) + \int_{V^c} 1 - Q_2 d\mu_2, \tag{3.6}$$

$$\int_U 1 - Q_1 d\mu_1 \le \iint_{U \times V} g_2 d(\mu_1 \times \mu_2) + \int_{V^c} Q_2 d\mu_2. \tag{3.7}$$

Simplifying the above, we obtain (2.5).

3.2. Proof of Theorem 2.2.

Proof of Theorem 2.2. The proof for Theorem 2.2 is similar to the proof for Theorem 2.1 but with the incentive compatibility requirements.

Step 1: Reduction to one function. By (i) and (ii) in Problem 2.2, we have the equality $Q_2(y) = \int_0^1 q_2(x,y)dx = 1 - \int_0^1 q_1(x,y)dx$. Since $q_1 + q_2 = 1$, if we find $q_1 : [0,1]^2 \mapsto [0, g_1]$, then $q_2 \ge 1 - g_1$. Similarly, if we find $q_2 : [0,1]^2 \mapsto [0, g_2]$, then $q_1 \ge 1 - g_2$. Thus, to satisfy Problem 2.2 (iii), it is necessary and sufficient to find $q_1 : [0,1]^2 \mapsto [0, g_1]$ and $q_2 : [0,1]^2 \to [0, g_2]$.

Take $\tilde{Q}_2(y) = 1 - Q_2(1-y)$. Clearly, $\tilde{Q}_2 : [0,1] \mapsto [0,1]$ is bounded, measurable, and nondecreasing. The condition $\int_0^1 (Q_1 + Q_2) = 1$ holds if and only if $\int_0^1 Q_1 = \int_0^1 \tilde{Q}_2$. Thus, to find $q_1 : [0,1] \to [0, g_1]$ that satisfies Problem 2.2 (iv), it is necessary and sufficient to find $f^1(x,y) : [0,1]^2 \mapsto [0, g_1]$ with the following properties:

$$\int_0^1 f^1(x,y)dy = Q_1 =: f_1^1(x), \quad \int_0^1 f^1(x,y)dx = 1 - Q_2(1-y) =: f_2^1(y) \tag{3.8}$$

for a.e. $x, y \in [0,1]$, and where f^1 is nondecreasing in each variable a.e. All that is known about the functions $f_1^1, f_2^1 : [0,1] \mapsto [0,1]$ is that they are nondecreasing and measurable with $\int_0^1 f_1 = \int_0^1 f_2$. Clearly, if such an f^1 exists, then $q_1(x,y) = f(x, 1-y)$ and $q_2(x,y) = 1 - f(x, 1-y)$ solves Problem 2.2; and conversely, if Problem 2.2 has a solution, then $f^1(x,y) = q_1(x, 1-y)$ solves (3.8).

Applying the same argument, finding the required $q_2 : [0,1]^2 \to [0, g_2]$ is the equivalent of finding f^2, such that

$$\int_0^1 f^2(x,y)dy = 1 - Q_1(1-x) =: f_1^2(x), \quad \int_0^1 f^2(x,y)dx = Q_2 =: f_2^2(y) \tag{3.9}$$

for a.e. $x, y \in [0,1]$, and where f^2 is nondecreasing in each variable.

Step 2: Derive (2.7) **using Lemma 3.2.** Take $f_1, f_2 : [0,1] \mapsto [0,1]$, where f_1, f_2 are measurable and nondecreasing, and where $\int_0^1 f_1 = \int_0^1 f_2$. The goal is to understand what additional assumptions of f_1, f_2 guarantee the existence of $f \in L^1\left([0,1]^2\right)$, such that

$$0 \le f \le g, g \in L^1([0,1]) \text{ is bounded and nonnegative;} \tag{3.10a}$$

$$\int_0^1 f(x,y)dy = f_1(x); \tag{3.10b}$$

$$\int_0^1 f(x,y)dx = f_2(y); \tag{3.10c}$$

$$f(x,y) \text{ is nondecreasing in both } x \text{ and } y, \text{ respectively,} \tag{3.10d}$$

for almost every $x, y \in [0,1]$.

Lemma 3.2. *The necessary and sufficient condition for the existence of such an f in* (3.10) *is*

$$\int_U f_1 \le \iint_{U\times V} g + \int_{V^c} f_2, \text{ for all measurable } U, V \subset [0,1]. \tag{3.11}$$

By applying Lemma 3.2 to conditions 3.8 and 3.9, respectively, one obtains the following two inequalities:

$$\int_U Q_1 \le \iint_{U\times V} g_1 + \int_{V^c} 1 - Q_2(1-y) \quad \text{for all measurable } U, V \subset [0,1]. \tag{3.12}$$

$$\int_U 1 - Q_1(1-x) \le \iint_{U\times V} g_2 + \int_{V^c} Q_2 \quad \text{for all measurable } U, V \subset [0,1]. \tag{3.13}$$

Simplifying the above, one obtains (2.7).

3.3. **Proof of Corollary 2.3.** Since in this case $g_1(x,y) = g_1(y)$, $\iint_{U\times V} g_1 = |U| \int_V g_1$. Let $s, t \in [0,1]$. When $|U| = 1-s$ is fixed, the left-hand side of (3.12) is maximized at $U = [s,1]$; meanwhile, the right-hand side of (3.12) is $|U|\int_V g_1 + |V^c| - \int_{V^c} Q_2(1-y)$. Since g_1 is decreasing and Q_2 is increasing, when $|V| = t$, the right-hand side is minimized at $V = [1-t, 1]$. Hence, without loss generality, one can replace U with $[s,1]$ and V with $[1-t,1]$ and then rewrite (3.12) as

$$\int_0^s Q_1 + \int_0^t Q_2 \ge t - (1-s)\int_{1-t}^1 g_1 \quad \text{for all } s,t \in [0,1]. \tag{3.14}$$

Similarly, since $g_2(x,y) = g_2(x)$, $\iint_{U\times V} g_2 = |V| \int_U g_2$. We fix $|U| = s$ and $|V| = 1-t$. Since Q_1 is increasing, the left-hand side of (3.13) is maximized when $U = [1-s, 1]$. Since g_2 is decreasing and Q_2 is increasing, the right-hand side of (3.13) is minimized at $U = [1-s,1]$ and $V = [t,1]$. Replacing U, V with $[1-s,1]$ and $[t,1]$, respectively, (3.13) becomes

$$\int_0^s Q_1 + \int_0^t Q_2 \ge s - (1-t)\int_{1-s}^1 g_2 \quad \text{for all } s,t \in [0,1]. \tag{3.15}$$

Condition (2.9) follows by taking conditions (3.14) and (3.15) together. □

4. Proof of Lemma 3.1

In this section, I prove Lemma 3.1. First, I show the necessity of (3.5) in step I. Second, I demonstrate the sufficiency of (3.5) for the discrete case in step II. Finally, I pass the discrete result to the continuous functions in (3.5) in step III.

4.1. **Step I. Proof of necessity.** It is indeed easy to see that (3.5) is necessary for the existence of such an f in (3.4). Here, first, we take the arbitrary, measurable $U, V \subset [0,1]$. Denote $\mu = \mu_1 \times \mu_2$. Then,

$$\int_U f_1 d\mu_1 \overset{(3.4b)}{=} \iint_{U\times[0,1]} f d\mu = \iint_{U\times V} f d\mu + \iint_{U\times V^c} f d\mu \overset{3.4a}{\le} \tag{4.1}$$

$$\le \iint_{U\times V} g d\mu + \iint_{[0,1]\times V^c} f d\mu \overset{(3.4c)}{\le} \iint_{U\times V} g d\mu + \int_{V^c} f_2 d\mu_2, \tag{4.2}$$

where V^c denotes the complement of V in $[0,1]$. □

4.2. **Step II. Proof of sufficiency for the discrete case.** To understand why (3.5) is sufficient, I consider a discrete "version" of the problem here in step II. Later, in step III, I pass the solution for the discrete problem to a limit in order to obtain (3.5).

For step II, we obtain the sufficiency condition for the existence of a matrix with given row sums and column sums (marginals), which is dominated by another given matrix.

In what follows, I denote $[n] = \{1, ..., n\}$ for any positive integer $n \geq 1$.

Lemma 4.1. *(Kellerer, 1961, Satz 3.1) Let the numbers p_i, q_j, t_{ij}, where $(i, j) \in [n] \times [m]$, be such that*

(1) $t_{ij} \geq 0$ *for all* $(i, j) \in [n] \times [m]$*;*
(2) $\sum_{i\in[n]} p_i = \sum_{j\in[m]} q_j$*;*
(3)
$$\sum_{i\in A} p_i \leq \sum_{(i,j)\in A\times B} t_{ij} + \sum_{j\in B^c} q_j$$
for all $(A, B) \subset [n] \times [m]$*, where* $\sum_{i\in A} p_i = 0$ *if* $A = \emptyset$*, and where* $B^c = [n] \setminus B$ *denotes the complement in* $[m]$*.*

Then there exists a matrix, $\{s_{ij}\}_{(i,j)\in[n]\times[m]}$*, such that* $0 \leq s_{ij} \leq t_{ij}$*,* $\sum_{j\in[m]} s_{ij} = p_i$*, and* $\sum_{i\in[n]} s_{ij} = q_j$ *for all* $i \in [n], j \in [m]$*.*

Remark 4.2. *Note that Lemma 4.1 (3) implies* $q_j \geq 0$ *simply by taking* $A = \emptyset$ *and* $B^c = \{j\}$*, and also that* $(2) - (3)$ *implies* $p_i \geq 0$ *by taking* $B^c = \emptyset$ *and* $A = \{i\}$*.*

Remark 4.3. *In particular, if* $t_{ij} = 1$ *for all* $(i, j) \in A\times B$*, then condition* (3) *becomes* $\sum_{i\in A} p_i \leq |A||B| + \sum_{j\in B^c} q_j$*, where* $|A|, |B|$ *denote the cardinalities of* A, B*.*

In fact, one can easily see that Lemma 4.1 (3) is also a necessary condition for the existence of the matrix $\{s_{ij}\}_{i,j=1}^{n,m}$. The proof proceeds precisely in the same way as the proof in the continuous case in step I. However, this proof is not needed for the purposes of this step.

Proof of Lemma 4.1. Proof of Lemma 4.1 is achieved by induction on $n + m$. First, consider the base case where $m + n = 2$. If either $n = 0$ or $m = 0$, then by hypothesis (2), $p_i = q_j = 0$ for all i, j, and therefore, $s_{ij} = 0$ for all i, j solves the problem. If $m = n = 1$, then $p_1 = q_1$ by (2). Since $p_\emptyset = q_\emptyset = 0$, (3) requires $p_1 \leq t_{1,1} + 0$, $q_1 \leq t_{1,1} + 0$. If this is satisfied, then $s_{1,1} = p_1 = q_1$ is the solution.

Next, I briefly summarize the induction argument. I start by reducing the values of each $t_{i,j}$ slightly, as long as the 2^{n+m} linear inequalities in hypothesis (3) are not violated. Here, the following two scenarios can occur.

Case 1. All $t_{i,j}$ become zero. In this case, (3) implies that $\sum_{i\in A} p_i \leq \sum_{j\in B^c} q_j$ for all $(A, B) \subseteq [n] \times [m]$. Choosing $B = [m]$, $A = [n]$, then $\sum_{i\in[n]} p_i \leq 0$. Since p_i is nonnegative, then $p_i = 0$ for all $i \in [n]$. By $\sum_{i\in[n]} p_i = \sum_{j\in[m]} q_j$, one obtains $q_j = 0$ for all $j \in [m]$. Thus, $s_{ij} \equiv 0$ solves the problem.

Case 2. There exists $(Y_1, Y_2) \subseteq [n] \times [m]$ and $(i_0, j_0) \in (Y_1, Y_2)$, such that $t_{i_0 j_0} > 0$ and

$$\sum_{i\in Y_1} p_i = \sum_{(i,j)\in Y_1\times Y_2} t_{ij} + \sum_{j\in Y_2^c} q_j. \tag{4.3}$$

Without loss of generality, one can assume that $Y_1 = \{1, ..., n_1\}, Y_2 = \{m_1, ..., m\}$ for some integers n_1, m_1, with $1 \leq n_1 \leq n$ and $1 \leq m_1 \leq m$ (see Figure 1 for an illustration).

Note that if the required $s_{i,j}$ exists, then

$$\begin{aligned}\sum_{i\in Y_1} p_i &= \sum_{(i,j)\in Y_1\times Y_2} s_{ij} + \sum_{(i,j)\in Y_1\times Y_2^c} s_{ij} \leq \sum_{(i,j)\in Y_1\times Y_2} s_{ij} + \sum_{(i,j)\in Y_1\times Y_2^c} s_{ij} + \sum_{(i,j)\in Y_1^c\times Y_2^c} s_{ij} \\ &= \sum_{(i,j)\in Y_1\times Y_2} s_{ij} + \sum_{j\in Y_2^c} q_j \leq \sum_{(i,j)\in Y_1\times Y_2} t_{ij} + \sum_{j\in Y_2^c} q_j.\end{aligned}$$

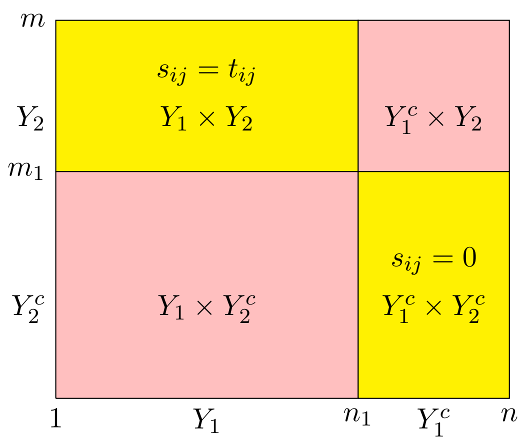

FIGURE 1. Sets Y_1, Y_1^c, Y_2, Y_2^c

Equality (4.3) implies that $\sum_{(i,j)\in Y_1^c\times Y_2^c} s_{ij} = 0$ and $\sum_{(i,j)\in Y_1\times Y_2} s_{ij} = \sum_{(i,j)\in Y_1\times Y_2} t_{ij}$ must hold. Since $0 \le s_{ij} \le t_{ij}$, one can conclude that $s_{ij} = 0$ for $(i,j) \in Y_1^c \times Y_2^c$ and that $s_{ij} = t_{ij}$ for $(i,j) \in Y_1 \times Y_2$.

In order to find s_{ij} on the rectangle $Y_1 \times Y_2^c$ (and $Y_1^c \times Y_2$), ideally, we would apply the induction assumption to the smaller rectangles; however, the obstacle is that (2) and (3) might not hold on $Y_1 \times Y_2^c$ (notice that the complement of a subset $A \subset Y_1$ in Y_1 differs from its complement in $[n]$). To solve this issue, we must modify the numbers p_i, q_j, t_{ij} in a certain way so that the new numbers p_i', q_j', t_{ij}' will satisfy the induction assumption for these smaller rectangles. After this modification, one can then "glue" all the solutions for the four different rectangles into one solution on $[n] \times [m]$. By doing so, we find that this solution satisfies the required conditions on the entire domain $[n] \times [m]$.

Lemma 4.4. *(Kellerer, 1961, Satz. 3.2) Suppose that* (1), (2), (3) *hold for* $\{p_i\}_{i=1}^n, \{q_j\}_{j=1}^m, \{t_{ij}\}_{i,j=1}^{n,m}$, *and that there exists* $Y_1 \times Y_2$, *such that (4.3) holds. Now, consider the following real numbers* $\{p_i'\}_{i=1}^n$, $\{q_j'\}_{j=1}^m$, *and* $\{t_{ij}'\}_{i,j=1}^{n,m}$:

$$t_{ij}' = \begin{cases} 0 & \text{on } (Y_1, Y_2) \cup (Y_1^c, Y_2^c) \\ t_{ij} & \text{otherwise;} \end{cases} \tag{4.4a}$$

$$p_i' = p_i - \chi_{Y_1}(i) \cdot \sum_{Y_2} t_{ij}; \tag{4.4b}$$

$$q_j' = q_j - \chi_{Y_2}(j) \cdot \sum_{Y_1} t_{ij}. \tag{4.4c}$$

Then (1), (2), (3) *hold for* $\{p_i'\}_{i=1}^n$, $\{q_j'\}_{j=1}^m$, $\{t_{ij}'\}_{i,j=1}^{n,m}$ *on* $Y_1 \times Y_2^c$ *and on* $Y_1^c \times Y_2^c$.

Proof of Lemma 4.4. (i) $t_{i,j}' \ge 0$ on $[n] \times [m]$ follows from the fact that t_{ij} is nonnegative on $[n] \times [m]$. Therefore, (1) holds for t_{ij}'.

(ii) The following equalities show that (2) holds for p_i', q_j', t_{ij}' on $Y_1 \times Y_2^c$ and on $Y_1^c \times Y_2$.

$$\sum_{i\in Y_1} p_i' \overset{4.4b}{=} \sum_{i\in Y_1} p_i - \sum_{Y_1\times Y_2} t_{ij} \overset{4.3}{=} \sum_{j\in Y_2^c} q_j \overset{4.4c}{=} \sum_{j\in Y_2^c} q_j',$$
$$\sum_{i\in Y_1^c} p_i' \overset{4.4b}{=} \sum_{i\in Y_1^c} p_i \overset{4.3}{=} \sum_{j\in Y_2} q_j - \sum_{Y_1\times Y_2} t_{i,j} \overset{4.4c}{=} \sum_{j\in Y_2} q_j'.$$

(iii) First, take $A \times B \subseteq Y_1 \times Y_2^c$. By (3), $\sum_A p_i \le \sum_{A\times(Y_2\cup B)} t_{ij} + \sum_{Y_2^c\setminus B} q_j$. By splitting the domain $A \times (Y_2 \cup B)$ into the disjoint union $(A \times Y_2) \cup (A \times B)$ and by separating the summations

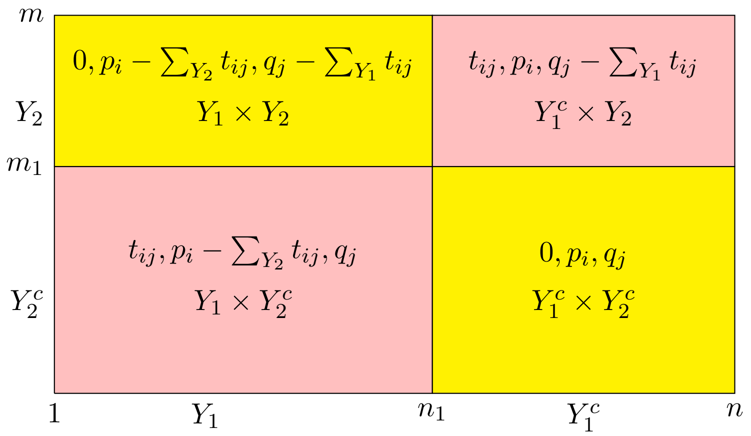

FIGURE 2. Modifications

on the disjoint sets and rearranging them, one obtains the following:

$$\sum_A p_i - \sum_{A \times Y_2} t_{ij} \leq \sum_{A \times B} t_{ij} + \sum_{Y_2^c \setminus B} q_j.$$

Because $\sum_A p_i' = \sum_A p_i - \sum_{A \times Y_2} t_{ij}$ by (4.4b), $\sum_{Y_2^c \setminus B} q_j' = \sum_{Y_2^c \setminus B} q_j$ by (4.4c), and $\sum_{A \times B} t_{ij}' = \sum_{A \times B} t_{ij}$ by (4.4a), the above is equivalent to

$$\sum_A p_i' \leq \sum_{A \times B} t_{ij}' + \sum_{Y_2^c \setminus B} q_j'. \tag{4.5}$$

Next, take $A \times B \subseteq Y_1^c \times Y_2$. By (3), $\sum_{A \cup Y_1} p_i \leq \sum_{(A \cup Y_1) \times B} t_{ij} + \sum_{Y_2^c \cup (Y_2 \setminus B)} q_j$. Subtracting equality (4.3) from this inequality, one gets the following:

$$\sum_{A \cup Y_1} p_i - \sum_{Y_1} p_i \leq \sum_{(A \cup Y_1) \times B} t_{ij} + \sum_{Y_2^c \cup (Y_2 \setminus B)} q_j - \sum_{Y_1 \times Y_2} t_{ij} - \sum_{j \in Y_2^c} q_j.$$

Note that $Y_1 \times Y_2 = (Y_1 \times B) \cup (Y_1 \times (Y_2 \setminus B))$ and that all the unions are disjoint. The above inequality can be reduced to obtain the following:

$$\sum_A p_i \leq \sum_{A \times B} t_{ij} + \sum_{Y_2 \setminus B} q_j - \sum_{Y_1 \times (Y_2 \setminus B)} t_{ij}.$$

Because $\sum_A p_i' = \sum_A p_i$ by (4.4b), $\sum_{Y_2 \setminus B} q_j' = \sum_{Y_2 \setminus B} q_j - \sum_{Y_1 \times (Y_2 \setminus B)} t_{ij}$ by (4.4c), and $\sum_{A \times B} t_{ij}' = \sum_{A \times B} t_{ij}$ by (4.4a), the above inequality can be written as follows:

$$\sum_A p_i' \leq \sum_{A \times B} t_{ij}' + \sum_{Y_2 \setminus B} q_j'. \tag{4.6}$$

(4.5) and (4.6) are the same as Lemma 4.1 (3) when the latter is restricted to the domains $Y_1 \times Y_2^c$ and $Y_1^c \times Y_2$, respectively, for the new variables $\{p_i'\}_{i=1}^n$, $\{q_j'\}_{j=1}^m$, $\{t_{ij}'\}_{i,j=1}^{n,m}$. □

Inductive step. Let $m + n > 2$. Here, I show that if the required $\{s_{ij}\}_{i,j=1}^{m',n'}$ exists for $m' + n' < m + n$, then the required $\{s_{ij}\}_{i,j=1}^{m,n}$ also exists. Since $|Y_1| + |Y_2^c| = m_1 + n_1 < m + n$ and $|Y_1^c| + |Y_2| < m + n - m_1 - n_1$, then we can apply the induction hypothesis on the domain $Y_1 \times Y_2^c$ and $Y_1^c \times Y_2$. That is, there exists $\{s_{ij}\}$ on the sets $Y_1 \times Y_2^c$ and $Y_1^c \times Y_2$, such that $0 \leq s_{ij} \leq t_{ij}$ for all $(i,j) \in (Y_1 \times Y_2^c)$ and $(i,j) \in (Y_1^c \times Y_2)$, and

$$\text{on } (i,j) \in Y_1 \times Y_2^c: \quad \sum_{j \in Y_2^c} s_{ij} = p_i' = p_i - \sum_{j \in Y_2} t_{ij}, \quad \sum_{i \in Y_1} s_{ij} = q_j' = q_j,$$

$$\text{on } (i,j) \in Y_1^c \times Y_2: \quad \sum_{j \in Y_2} s_{ij} = p_i' = p_i, \quad \sum_{i \in Y_1^c} s_{ij} = q_j' = q_j - \sum_{i \in Y_1} t_{ij}.$$

Combining the above with the result that $s_{ij} = t_{ij}$ on $Y_1 \times Y_2$ and $s_{ij} = 0$ on $Y_1^c \times Y_2^c$, one finds that $0 \le s_{ij} \le t_{ij}, (i,j) \in [n] \times [m]$; and also that

$$\begin{aligned}
i \in Y_1 : \sum_{j\in[m]} s_{ij} &= \sum_{j\in Y_2} s_{ij} + \sum_{j\in Y_2^c} s_{ij} = \sum_{j\in Y_2} t_{ij} + p_i - \sum_{j\in Y_2} t_{ij} = p_i;\\
i \in Y_1^c : \sum_{j\in[m]} s_{ij} &= \sum_{j\in Y_2} s_{ij} + \sum_{j\in Y_2^c} s_{ij} = p_i + 0 = p_i;\\
j \in Y_2 : \sum_{i\in[n]} s_{ij} &= \sum_{i\in Y_1} s_{ij} + \sum_{i\in Y_1^c} s_{ij} = \sum_{i\in Y_1} t_{ij} + q_j - \sum_{i\in Y_1} t_{ij} = q_j;\\
j \in Y_2^c : \sum_{i\in[n]} s_{ij} &= \sum_{i\in Y_1} s_{ij} + \sum_{i\in Y_1^c} s_{ij} = q_j + 0 = q_j.
\end{aligned}$$

Therefore, we see that s_{ij} satisfies the required conditions for the entire set $[n] \times [m]$. □

To summarize, thus far, we have completed the proof that a matrix $\{s_{i,j}\}_{(i,j)\in[n]\times[m]}$ exists if and only if $\{p_i, q_j, t_{ij}\}$ satisfies the hypothesis in Lemma 4.1.

4.3. **Step III. Proving (3.5) using the discrete case.** We denote the intervals $R_1^i = \left[\frac{i-1}{n}, \frac{i}{n}\right], R_2^j = \left[\frac{j-1}{n}, \frac{j}{n}\right]$ and the rectangles $R^{ij} = R_1^i \times R_2^j$ for $i, j \in [n]$.

We take the functions f_1, f_2, g from (3.5) and define the following numbers:

$$g_{ij} := \int_{R^{ij}} gd(\mu_1 \times \mu_2), \quad f_1^i := \int_{R_1^i} f_1 d\mu_1, \quad f_2^j := \int_{R_2^j} f_2 d\mu_2. \tag{4.7}$$

Conditions (1), (2), and (3) in Lemma 4.1 hold for the numbers f_1^i, f_2^j, and g_{ij}. Lemma 4.1 (1) holds because $g_{ij} \ge 0$. Lemma 4.1 (2) holds because

$$\sum_{i=1}^n f_1^i = \sum_{i=1}^n \int_{R_1^i} f_1 d\mu_1 = \int_{[0,1]} f_1 d\mu_1 = \int_{[0,1]} f_2 d\mu_2 = \sum_{j=1}^n \int_{R_2^j} f_2 d\mu_2 = \sum_{j=1}^n f_2^j.$$

To verify Lemma 4.1 (3), for any $A, B \subset [n] \times [n]$, let $U = \cup_{i\in A} R_1^i$, $V^c = \cup_{j\in B^c} R_2^j$, and let $U \times V = \cup_{(i,j)\in A\times B} R^{ij}$. Then,

$$\sum_A f_1^i = \sum_A \int_{R_1^i} f_1 d\mu_1 = \int_U f_1 d\mu_1, \quad \sum_{B^c} f_2^j = \sum_{B^c} \int_{R_2^j} f_2 d\mu_2 = \int_{V^c} f_2 d\mu_2$$

and

$$\sum_{(i,j)\in(A\times B)} g_{ij} = \sum_{A\times B} \iint_{R^{ij}} gd(\mu_1 \times \mu_2) = \iint_{U\times V} gd(\mu_1 \times \mu_2).$$

By (3.5), (3) holds. Therefore, by Lemma 4.1, there exists a matrix $\{F_{i,j}\}_{i,j=1}^n$, such that $0 \le F_{i,j} \le g_{ij}$ for all $i \in A, j \in B$, and $\sum_{i=1}^n F_{i,j} = f_2^j$, $\sum_{j=1}^n F_{i,j} = f_1^i$, $\forall i, j \in [n]$.

Next, I pass F_{ij} to the *weak** limit to find the sufficient condition for (3.5). Letting $\mu = \mu_1 \times \mu_2$, I define the following functions:

$$F_n(x,y) = \sum_{i,j=1}^n \frac{F_{i,j}}{\mu(R^{i,j})} \mathbb{1}_{R^{ij}}(x,y); \tag{4.8a}$$

$$G_n(x,y) = \sum_{i,j=1}^n \frac{g_{i,j}}{\mu(R^{i,j})} \mathbb{1}_{R^{ij}}(x,y) \to g(x,y); \tag{4.8b}$$

$$F_1^n(x) = \sum_{i=1}^n \frac{f_1^i}{\mu_1(R_1^i)} \mathbb{1}_{R_1^i}(x) \to f_1(x), x \in [0,1]; \tag{4.8c}$$

$$F_2^n(x) = \sum_{j=1}^n \frac{f_2^j}{\mu_2(R_2^j)} \mathbb{1}_{R_2^j}(y) \to f_2(y), y \in [0,1]. \tag{4.8d}$$

The convergence results in (4.8b), (4.8c), and (4.8d) are found by the Lebesgue differentiation theorem (see, e.g., Benedetto & Czaja, 2010, Theorem 8.4.6). Next, I show that the *weak** limit of F_n exists and that it satisfies (3.4).

Lemma 4.5. *There exists a subsequence* $\{F_{n_k}\}$ *of* $\{F_n\}$*, such that* $F_{n_k} \overset{w^*}{\to} F$ *for some* $F \in L^1\left([0,1]^2, \mu_1 \times \mu_2\right)$*, and the* w^**-limit* F *satisfies* (3.4)*, i.e.,*

1. $0 \le F(x,y) \le g(x,y)$ *a.e.,*
2. $f_2(y) = \int_{[0,1]} F(x,y) d\mu_1(x)$ *a.e.,*
3. $f_1(x) = \int_{[0,1]} F(x,y) d\mu_2(y)$ *a.e.*

Proof of Lemma 4.5. By definition, $0 \le F_n(x,y) \le G_n(x,y)$, and $0 \le G_n(x,y)$ is bounded. Without loss of generality, let $0 \le G_n(x,y) \le 1$. Therefore, the sequence $\{F_n(x,y)\}$ belongs to a unit ball of $L^\infty([0,1]^2, \mu)$, where $\|F_n(x,y)\|_{L^\infty([0,1]^2,\mu)} \le 1$ for all $n \ge 1$. By the Banach-Alaoglu theorem, there exists a subsequence $\{F_{n_k}\}, k = 1, 2, ...,$ such that

$$F_{n_k}(x,y) \overset{w^*}{\to} F(x,y), F \in L^\infty([0,1]^2, \mu).$$

Next, we verify the properties (1) through (3).

(1) We know that on $[0,1]$, $0 \le F_{n_k} \le G_{n_k}$ a.e., and that $G_{n_k} \to g$ a.e., G_{n_k} is bounded. Taking any measurable set $U \subseteq [0,1]^2$, $|U| > 0$. The following must then hold:

$$\int_U F_{n_k} d\mu \le \int_U G_{n_k} d\mu \to \int_U g d\mu.$$

Let $h = \mathbb{1}_U \in L^1([0,1]^2, \mu)$. For the sake of contradiction, assume $F > g + \epsilon$ on U for some $\epsilon > 0$. Therefore,

$$\lim_{k\to\infty} \int_U F_{n_k} d\mu = \lim_{k\to\infty} \int_{[0,1]^2} F_{n_k} \cdot h d\mu \overset{F_{n,k} \overset{w^*}{\to} F}{=} \int_{[0,1]^2} F \cdot h d\mu = \int_U F d\mu.$$

It would then follow that $\int_U F d\mu \le \int_U g d\mu$. This contradicts $F > g + \epsilon$ on U.

(2) Let $l = \mathbb{1}_{\{f_2(\tilde{y}) - \int_{[0,1]} F(x,\tilde{y}) d\mu_1(x) < 0\}}(y)$. Then, $l \in L_1([0,1]^2, \mu_1)$. Since $F_{n_k} \overset{w^*}{\to} F$,

$$\lim_{k\to\infty} \int_{[0,1]^2} F_{n_k}(x,y) \cdot l(y) d\mu = \int_{[0,1]^2} F(x,y) \cdot l(y) d\mu.$$

By Fubini's theorem, since $d\mu = d\mu_1 \times d\mu_2$, we can write the above as

(4.9)
$$\lim_{k\to\infty} \int_{[0,1]} \left(\int_{[0,1]} F_{n_k}(x,y) d\mu_1(x) \right) \cdot l(y) d\mu_2(y) = \int_{[0,1]} \left(\int_{[0,1]^2} F(x,y) d\mu_1(x) \right) \cdot l(y) d\mu_2(y).$$

Moreover,

$$\int_{[0,1]} F_{n_k}(x,y) d\mu_1(x) \overset{(4.8a)}{=} \int_{[0,1]} \sum_{i,j=n_1}^{n_k} \frac{F_{i,j}}{\mu(R^{i,j})} \mathbb{1}_{R^{i,j}}(x,y) d\mu_1(x) =$$
$$= \sum_{i=n_1}^{n_k} \int_{R_1^i} \sum_{j=n_1}^{n_k} \frac{F_{i,j}}{\mu_1(R_1^i)\mu_2(R_2^j)} \mathbb{1}_{R_1^j}(y) d\mu_1(x) =$$
$$= \sum_{i=n_1}^{n_k} \sum_{j=n_1}^{n_k} \frac{F_{i,j}}{\mu_1(R_1^i)\mu_2(R_2^j)} \mathbb{1}_{R_1^j}(y) \mu_1(R_1^i) =$$
$$= \sum_{j=n_1}^{n_k} \frac{\sum_{i=n_1}^{n_k} F_{i,j}}{\mu_2(R_2^j)} \mathbb{1}_{R_1^j}(y) \overset{(4.8d)}{=} \sum_{j=n_1}^{n_k} \frac{f_2^j}{\mu_2(R_2^j)} \mathbb{1}_{R_1^j}(y) = F_2^{n_k}(y).$$

Therefore, the left-hand-side of (4.9) satisfies the following:

$$LHS = \lim_{k\to\infty} \int_{[0,1]} F_2^{n_k}(y) \cdot l(y) d\mu_2(y) = \int_{[0,1]} \lim_{k\to\infty} F_2^{n_k}(y) \cdot l(y) d\mu_2(y)$$
$$= \int_{[0,1]} f_2(y) \cdot l(y) d\mu_2(y)$$

since the limit can be taken inside of the integration by the Lebesgue dominated convergence theorem. By linearity of integration, (4.9) implies

$$\int_{[0,1]} \left(f_2(y) - \int_{[0,1]} F(x,y) d\mu_1(x) \right) \cdot l(y) d\mu_2(y) = 0.$$

Hence, $l(y) \equiv 0$ on $[0,1]$, and thus, $f_2(y) - \int_{[0,1]} F(x,y)d\mu_1(x) \geq 0$ a.e. Now, we let $l = \mathbb{1}_{\{f_2(\tilde{y}) - \int_{[0,1]} F(x,\tilde{y})d\mu_1(x) > 0\}}(y)$ and repeat the same process. From this work, we conclude that $f_2(y) - \int_{[0,1]} F(x,y)d\mu_1(x) \leq 0$ a.e. Therefore, $f_2(y) = \int_{[0,1]} F(x,y)d\mu_1(x)$ a.e.

(3) The proof is the same as in (2), with x, y reversed, and with f_2 replaced by f_1.

□

5. Proof of Lemma 3.2

The proof for Lemma 3.2 uses several modifications of the proof for Lemma 3.1 to accommodate the incentive compatibility.

5.1. **Step I. Sufficiency of 3.11.** The proof for sufficiency is the same as that in the proof of Lemma 3.1, with $\mu_1 = \mu_2 = \lambda$, the Lebesgue measure.

5.2. **Step II. Discrete case.** For step II, we obtain the sufficiency condition for the existence of a matrix with given increasing row sums and column sums (marginals), which is dominated by another given matrix. Additionally, the entries of the desired matrix must increase monotonically with the row and column indexes, respectively. That is, for the matrix $A = [a_{i,j}]_{i,j=1}^n$, we must have $a_{ij} \leq a_{ik}$ for all $1 \leq j \leq k \leq n$ for any $i = 1, ..., n$, and $a_{ij} \leq a_{kj}$ for all $1 \leq i \leq k \leq n$ for any $j = 1, ..., n$.

Lemma 4.1 gives us the condition for the existence of a dominated matrix for given marginals. When the marginals are both increasing, the following lemma shows that there exists a matrix that satisfies the required monotonicity property.

Lemma 5.1. *(Gutmann et al., 1991, Theorem 6) Let $\{s_{ij}\}_{i=1}^{n,m}$ be an $n \times m$ matrix with $0 \leq s_{ij} \leq t_{ij}$, having nondecreasing row sums and column sums. In other words,*

$$p_i = \sum_{j=1}^{m} s_{ij}, i \in [n] \quad \text{and} \quad q_j = \sum_{i=1}^{n} s_{ij}, j \in [m] \tag{5.1}$$

satisfy $p_i \leq p_{i+1}$, $i \in [n-1]$, and $q_j \leq q_{j+1}$, $i \in [m-1]$. In this case, there exists another $n \times m$ matrix $\{\psi_{ij}\}_{i,j=1}^{n,m}$ with $0 \leq \psi_{ij} \leq t_{ij}$, such that

$$p_i = \sum_{j=1}^{m} \psi_{ij}, i \in [n]; \quad q_j = \sum_{i=1}^{n} \psi_{ij}, j \in [m]; \tag{5.2a}$$

$$\psi_{ij} \leq \psi_{i+1,j} \text{ and } \psi_{ij} \leq \psi_{i,j+1} \text{ for all } (i,j) \in [n-1] \times [m-1]. \tag{5.2b}$$

Proof. For this proof, I adapt Theorem 6 from Gutmann et al. (1991). Let $\{\psi_{ij}\}_{i,j=1}^{n,m}$ be the unique $n \times m$ matrix with $0 \leq \psi_{ij} \leq 1$, which satisfies (5.2a), such that $\sum_{(i,j)\in[n\times m]} (\psi_{i,j})^2$ is as small as possible. It is sufficient to show that $\{\psi_{ij}\}_{i,j=1}^{n,m}$ satisfies (5.2b).

Then, suppose instead that for some $(i,j) \in [n-1] \times [m-1]$:

$$0 \leq \psi_{i+1,j} < \psi_{ij} \leq t_{ij}. \tag{5.3}$$

Since $\sum_{j=1}^{m} \psi_{i,j} = p_i \leq p_{i+1} = \sum_{j=1}^{m} \psi_{i+1,j}$, there exists $1 \leq k \leq m$, for which

$$0 \leq \psi_{ik} < \psi_{i+1,k} \leq t_{i+1,k}. \tag{5.4}$$

By the inequalities in (5.3) and (5.4), $\psi_{i+1,j}$ and ψ_{ik} cannot be t_{ij} and $t_{i+1,k}$, while ψ_{ij} and $\psi_{i+1,k}$ cannot be 0. Therefore, for a small enough $\epsilon > 0$, one can increase $\psi_{i+1,j}$ and ψ_{ik} by ϵ, and one can decrease ψ_{ij} and $\psi_{i+1,k}$ by ϵ:

$$\begin{matrix} \psi_{ij} \downarrow & \cdots & \psi_{ik} \uparrow \\ \psi_{i+1,j} \uparrow & \cdots & \psi_{i+1,k} \downarrow \end{matrix}$$

We then arrive at a new matrix, $\{\psi'_{ij}\}_{i,j=1}^{n,m}$ with $0 \leq \psi'_{ij} \leq t_{ij}$ with the same row and column sums; however, $\sum_{(i,j)\in[n]\times[m]}(\psi'_{ij})^2 - \sum_{(i,j)\in[n]\times[m]}(\psi_{ij})^2 = 2(\psi_{i+1,j} - \psi_{ij} + \psi_{ik} - \psi_{i+1,k}) + 4\epsilon^2 < 0$, which contradicts the minimality of $\sum_{i,j=1}^{n,m}(\psi_{ij})^2$. ☐

To summarize, by using Lemmas 4.1 and 5.1, we completed the proof that a matrix $\{s_{i,j}\}_{(i,j)\in[n]\times[m]}$ whose column entries and row entries both increase exists if and only if $\{p_i, q_j, t_{ij}\}$ satisfies the hypothesis in Lemma 4.1, as well as the additional condition that p_i, q_i is nondecreasing.

5.3. **Step III. Proving** (3.11) **using the discrete case.** The proof of (3.11) is almost the same as that for (3.5) with a few modifications. First, we take $\mu_1 = \mu_2 = \lambda$, the Lebesgue measure. When defining the numbers g_{ij}, f_1^i, f_2^j, we take the functions f_1, f_2, g from (3.11) instead of from (3.5). Thus f_1^i, f_2^j are increasing since f_1, f_2 are increasing.

Conditions (1), (2), and (3) in Lemma 4.1 hold for the numbers f_1^i, f_2^j, and g_{ij} by the same argument as that in the previous subsection. By Lemma 5.1, without loss of generality, one can assume that $F_{i,j}$ will increase in both i and j since f_1^i, f_2^j are increasing.

By Lemma 4.5, the *weak** limit of F_n exists, and it satisfies (3.10) (1) (2) (3). Furthermore, as $f_1 = \int_0^1 F$ increases in x and $f_2 = \int_0^1 F$ increases in y, it is clear that F must increase in both x, y independently. [8]

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[8] $F(x,y) \in L^\infty([0,1]^2, \mu)$ increasing on x and y means that for $C, D \subset [0,1]^2$, where C, D are disjoint with equal measure, and where D is northeast of C, then $\int_C F \leq \int_D F$.

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